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Star-Deformation of the Infinite Oscillator Algebra and the Realization of Both q-Deformed Centerless Virasoro and $SU_q(2)$ Algebras

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This paper describes the infinite oscillator algebra deformation along the lines of the star-deformation method. Realization of a q-deformed centerless Virasoro algebra and $SU_q(2)$ algebra are obtained.

1. INTRODUCTION

Theoretical physics has recently given much attention to quantum groups (Drinfeld, 1987; Jimbo, 1985, 1989; Manin, 1988).

The interest in quantum groups arose almost simultaneously in statistical mechanics and in conformal theories, in solid state and in the study of topologically nontrivial solutions of nonlinear equations, so that the research in quantum groups grew along parallel lines from physical as well as mathematical problems.

As is well known, quantum groups can be seen as a noncommutative generalization of topological space which has a group structure. Such a structure induces an abelian Hopf algebra (Abe, 1980) structure on the algebra of functions defined on the group. Quantum groups are defined then as a nonabelian Hopf algebra (Takhtajan, 1989). A way to generate them consists in deforming the product of a Hopf algebra into a nonabelian one (*-product) using the so-called quantization by deformation (Bayen *et al.*, 1978a, b) or star-deformation procedure.

With the discovery of quantum groups, q-generalizations of the harmonic oscillator have recently been the center of attention (Biedenharn, 1989; Mac-

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farlane, 1989; Chaichian *et al.*, 1990, 1991; Chakrabarti and Jagannathan, 1991; Fairly and Zachos, 1991). There are different formulations of these generalizations, the so-called q-oscillators. Strict constraints on these generalizations come from the multidimensionality of the oscillator. The deformed oscillator algebra is defined as an associative algebra generated by the generators, a, a^+, and N, and the defining relations are obtained by deformation of the canonical commutation relations.

Virasoro algebra (Gel'fand and Fuchs, 1968; Virasoro, 1970), which has been efficient in a great number of both physical and mathematical applications, is a one-dimensional central extension of the algebra of vector fields on a circle. On the other hand, the quantum group $Su_q(2)$ was first introduced by Skylanin (1982) and independently by Kulich and Reshetikhin (1983) in their work on the Yang–Baxter equation.

In contrast to previous work (Mansour, 1997), where an h-deformation of a Lie algebra is given in terms of an invariant star product on the corresponding Lie group, the purpose of the present paper is to develop a new realization of a deformed centerless Visasoro algebra and a new realization of $Su_q(2)$ algebra using the deformation of the infinite oscillator algebra by the star-deformation method, where the star products used here, are a deformation of the product of the enveloping algebra.

2. THE INFINITE OSCILLATOR ALGEBRA AND REALIZATION OF BOTH CENTERLESS VIRASORO AND *SU*(2) ALGEBRAS

We consider an infinite number of harmonic oscillators with the following creation and annhilation operators $\{a_1^+, a_1^-, \ldots, a_n^+, a_n^-, \ldots, n \geq 1\}$. The infinite oscillator algebra $\mathcal{A}$ is the associative algebra generated by the operators $\{a_n^+, a_n^-, 1, \ldots, n \geq 1\}$ and the defining commutation relations

$$[a_n^-, a_m^+] = n\delta_{n,m} \tag{2.1}$$

$$[a_n^+, a_m^+] = 0 \tag{2.2}$$

$$[a_n^-, a_m^-] = 0 \tag{2.3}$$

$$[1, \text{everything}] = 0. \tag{2.4}$$

Let $N_k = (1/k)a_k^- a_k^+$ be the number operator of the kth oscillator and $N = \Sigma_k N_k$ be the number operator. From the relations (2.1)–(2.4) we obtain

$$[N_k, a_m^+] = \delta_{k,m}a_m^+ \tag{2.5}$$

$$[N_k, a_m^-] = -\delta_{k,m}a_m^- \tag{2.6}$$

$$[N_k, N_m] = 0. \tag{2.7}$$

Now if we consider the generators

$$J_n = \sum_{k \geq 1} \frac{1}{k} (a_k^-)^{n+1} a_k^+ \tag{2.8}$$

then using equation (2.1) we can easily show that

$$[J_n, J_m] = (n - m) J_{m+n} \tag{2.9}$$

i.e., the generators $\{J_n, n \in \mathbb{Z}\}$ satisfy the classical centerless Virasoro algebra.

It can be easily seen that the generators $\{J_0, J_{-1}, J_1\}$ satisfy the classical $SU(2)$ algebra:

$$[J_0, J_1] = -J_1 \tag{2.10}$$

$$[J_0, J_{-1}] = J_{-1} \tag{2.11}$$

$$[J_1, J_{-1}] = 2J_0 \tag{2.12}$$

3. STAR DEFORMATION OF THE INFINITE OSCILLATOR ALGEBRA AND REALIZATION OF BOTH DEFORMED CENTERLESS VIRASORO AND $SU_q(2)$ ALGEBRAS

In order to deform the infinite oscillator algebra $\mathcal{A}$, let us endow it with two star products $*_r$ and $*_l$ defined between their elements by

$$x_1 *_r x_2 = x_1 R(N) x_2 \tag{3.1}$$

$$x_1 *_l x_2 = x_1 L(N) x_2 \tag{3.2}$$

where $R(N)$ and $L(N)$ are smooth functions of the number operator N, which depend also on the deformation parameter q, and are given by

$$R(N) = \frac{[\Sigma_k N_k]_q}{\Sigma_k N_k} \frac{1}{[\Sigma_k N_k]_q - [\Sigma_k N_k - 1]_q} \tag{3.3}$$

$$L(N) = \frac{[\Sigma_k N_k + 1]_q}{\Sigma_k N_k + 1} \frac{1}{[\Sigma_k N_k + 2]_q - [\Sigma_k N_k + 1]_q} \tag{3.4}$$

and

$$[N]_q = \frac{q^N - 1}{q - 1}.$$

Remark that when $q = 1$, $R(N)$ and $L(N)$ tend to 1 simultaneously, and each of these star products reduces to the usual one.

The corresponding Moyal bracket is given by (Demosthenes, 1993)

$$[x_1, x_2]_{*_{rl}} = x_1 R(N) x_2 - x_2 L(N) x_1 \tag{3.5}$$

In this approach we define the star-deformed infinite oscillator algebra, denoted $({}_*\mathscr{A})$, as an associative algebra generated by $\{a_n^+, a_n^-, N, 1, \ldots, n \geq 1\}$ and the following relations:

$$[a_n^-, a_m^+]_{*rl} = n\delta_{n,m} \tag{3.6}$$

$$[a_n^+, a_m^+] = 0 \tag{3.7}$$

$$[a_n^-, a_m^-] = 0 \tag{3.8}$$

$$[N, a_m^+] = a_m^+ \tag{3.9}$$

$$[N, a_m^-] = -a_m^- \tag{3.10}$$

$$[1, \text{everything}] = 0. \tag{3.11}$$

Now we use the identities

$$f(N_m)a_n^+ = a_n^+ f(N_m + \delta_{n,m}) \tag{3.12}$$

$$f(N_m)a_n^- = a_n^- f(N_m - \delta_{n,m}) \tag{3.13}$$

which are equivalent to

$$f(N)a_n^+ = a_n^+ f(N + 1) \tag{3.14}$$

$$f(N)a_n^- = a_n^- f(N - 1) \tag{3.15}$$

and we rewrite equation (3.6) as

$$A_n^- A_m^+ - A_m^+ A_n^- = n\delta_{n,m}([N + 1]_q - [N]_q) \tag{3.16}$$

with

$$A_n^- = a_n^- \sqrt{\frac{[N]_q}{N}} = \sqrt{\frac{[N + 1]_q}{N + 1}}\, a_n^- \tag{3.17}$$

$$A_m^+ = \sqrt{\frac{[N]_q}{N}}\, a_m^+ = a_m^+ \sqrt{\frac{[N + 1]_q}{N + 1}}. \tag{3.18}$$

In the same way we have

$$[A_n^+, A_m^+] = 0$$

$$[A_n^-, A_m^-] = 0$$

$$[N, A_m^+] = A_m^+$$

$$[N, A_m^-] = -A_m^-$$

So, $({}_{*}\mathcal{A})$ is generated by the deformed generators $\{A_n^-, A_n^+, N, 1, \ldots, n \geq 1\}$ and the defining relations

$$A_n^- A_m^+ - A_m^+ A_n^- = n\delta_{n,m}([N+1]_q - [N]_q) \tag{3.19}$$

$$[A_n^+, A_m^+] = 0 \tag{3.20}$$

$$[A_n^-, A_m^-] = 0 \tag{3.21}$$

$$[N, A_m^+] = A_m^+ \tag{3.22}$$

$$[N, A_m^-] = -A_m^- \tag{3.23}$$

$$[1, \text{everything}] = 0. \tag{3.24}$$

Now, in a similar manner as in the undeformed case, if we consider the generators

$$J_n^q = q^{-n} \sum_{k\geq 1} \frac{1}{k} (A_k^-)^{n+1} A_k^+ \tag{3.25}$$

and introduce the deformed commutator $[\cdot,\cdot]_q$

$$[J_n^q, J_m^q]_q = J_n^q J_m^q - q^{(n-m)} J_m^q J_n^q \tag{3.26}$$

then we can easily show that the generators J_n^q satisfy the deformed Virasoro algebra:

$$[J_n^q, J_m^q]_q = q^{-n}([n]_q - [m]_q) J_{n+m}^q. \tag{3.27}$$

Remark that when $q = 1$, J_n^q reduces to J_n and $[n]_q = n$, and equation (3.27) reduces to equation (2.9).

Similarly to the classical case, the generators $\{J_0^q, J_1^q, J_{-1}^q\}$ generate $SU_q(2)$ algebra given by

$$\{J_0^q, J_1^q]_q = -J_1^q \tag{3.28}$$

$$\{J_0^q, J_{-1}^q]_q = \frac{1 - q^{-1}}{q - 1} J_{-1}^q \tag{3.29}$$

$$\{J_1^q, J_{-1}^q]_q = \frac{1 - q^{-2}}{q - 1} J_0^q \tag{3.30}$$

Remark that when $q \to 1$ the above deformed $SU(2)$ algebra reduces to the classical one, (2.10)–(2.12)

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